Appendix 3 Alternative method of factorization

This is based on the idea that the properties of a multiplicative function are strongly dictated by the values on the primes. So given multiplicative G look through known functions to find, if it exists, a multiplicative function g satisfying $g(p) = G(p)$ for all primes. Then we might think of G as a 'perturbation' of g and look for a multiplicative function f such that $G = g*f$. Note that on primes

$$\begin{aligned} G(p) &= (g*f)(p) = g(p)\, f(1) + g(1)\, f(p) \\ &= G(p) + f(p) \end{aligned}$$

since $g(p) = G(p)$. Hence $f(p) = 0$ for all primes. So we can think of f being, in turn, a perturbation of the zero function, $0(n) = 0$ for all n and thus f a 'small' function.

For an example consider Q_2 (even though we already know it's factorization). Because $Q_2(p) = 1$ and $1(p) = 1$ for all primes p that we may think that $Q_2 = 1 * f$ for some 'small' function f.

Important If given F which you suspect can be written as $1 * f$ for some 'simpler' f then, by Möbius inversion, $f = \mu * F$. If further F is multiplicative then f will be also and you need only calculate the values of f on prime powers.

$$\begin{aligned} f(p^r) &= \sum_{d|p^r} \mu(d)\, F\left(\frac{p^r}{d}\right) = \sum_{0\le k\le r} \mu\left(p^k\right) F\left(p^{r-k}\right) \\ &= \sum_{0\le k\le 1} \mu\left(p^k\right) F\left(p^{r-k}\right) \quad \text{since } \mu\left(p^k\right) = 0 \text{ for } k \ge 2, \\ &= F(p^r) - F\left(p^{r-1}\right), \end{aligned} \tag{19}$$

for $r \ge 1$. We will use this often so needs to be remembered.

Since f is multiplicative we will have $f(1) = 1$ and this need not be calculated. And when $r = 1$, $f(p^0) = f(1) = 1$ and so $f(p) = F(p) - 1$.

Example 3.42 *If $Q_2 = 1 * f_2$ describe f_2.*

Solution The function Q_2 is multiplicative so, by (19), we have

$$\begin{aligned} f_2(p^r) &= Q_2(p^r) - Q_2\left(p^{r-1}\right) \\ &= \begin{cases} 0-0 & \text{if } r-1 \geq 2 \\ 0-1 & \text{if } r-1 = 1 \\ 1-1 & \text{if } r-1 = 0 \end{cases} \\ &= \begin{cases} -1 & \text{if } r = 2, \\ 0 & \text{if } r \neq 2. \end{cases} \end{aligned}$$

Thus, writing $n = p_1^{a_1} p_2^{a_2} ... p_r^{a_r}$, we find that

$$f_2(n) = \begin{cases} (-1)^r & \text{if } a_1 = a_2 = ... = a_r = 2, \\ 0 & \text{otherwise.} \end{cases}$$

But $a_1 = a_2 = ... = a_r = 2$ means that $n = m^2$ with m square-free. But m square-free means that $\mu(m) = (-1)^r = f_2(n)$. And if $n = m^2$ but m is not square-free then $\mu(m) = 0 = f_2(n)$. Hence

$$f_2(n) = \begin{cases} \mu(m) & \text{if } n = m^2, \text{ i.e. } n \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

■

We may think of f_2 as 'small' since $f_2(p) = 0$ for all primes p and is thus a perturbation of the 0 function, $0(n) = 0$ for all n.

Notation In fact in the notes we use μ_2 in place of f_2, since this better represents the definition of the function. Thus

$$Q_2 = 1 * \mu_2.$$

We can use the decomposition of Q_2 to factor the Dirichlet series $D_{Q_2}(s)$.

Example 3.43

$$\sum_{n=1}^{\infty} \frac{Q_2(n)}{n^s} = \frac{\zeta(s)}{\zeta(2s)},$$

for $\operatorname{Re} s > 1$.

Proof

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{Q_2(n)}{n^s} &= D_{Q_2}(s) = D_{1*\mu_2}(s) \\
&= D_1(s)\, D_{\mu_2}(s) \qquad \text{by } (2) \\
&= \zeta(s) \sum_{n=1}^{\infty} \frac{\mu_2(n)}{n^s} \\
&= \zeta(s) \sum_{\substack{n=1 \\ n=m^2}}^{\infty} \frac{\mu(m)}{n^s} \\
&= \zeta(s) \sum_{m=1}^{\infty} \frac{\mu(m)}{m^{2s}} \\
&= \frac{\zeta(s)}{\zeta(2s)}.
\end{aligned}
$$

And this is valid wherever the final Riemann zeta functions are all absolutely convergent, i.e. $\operatorname{Re} s > 1$. ■

The method described in the notes turned this around; we first factor the Dirichlet Series to get the factorization of the arithmetic function.